

CONCENTRATION FOR MULTIDIMENSIONAL DIFFUSIONS AND THEIR BOUNDARY LOCAL TIMES

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ABSTRACT. We prove that probability laws of certain multidimensional semimartingales which includes time-inhomogeneous diffusions, under suitable assumptions, satisfy Quadratic Transportation Cost Inequality under the uniform metric. From this we derive concentration properties of Lipschitz functions of process paths that depend on the entire history. In particular, we estimate concentration of boundary local time of reflected Brownian motions on a polyhedral domain. We work out explicit applications of consequences of measure concentration for the case of Brownian motion with rank-based drifts.

1. INTRODUCTION

Consider the sample space (Ω, \mathcal{F}) where Ω is a metric space and \mathcal{F} is the associated Borel σ -algebra. We say that a probability measure μ on (Ω, \mathcal{F}) has the measure concentration property if the following statement holds. For any set $A \in \mathcal{F}$ such that $\mu(A) \geq 1/2$, one has $\mu(A_r)$ *very close* to one. Here $\mu(A_r)$ refers to the set of all points that are at a distance no larger than r from A . The closeness is typically expressed as a gaussian tail estimate in r . Moreover, for fixed r , the probability $\mu(A_r)$ tends to one exponentially fast in the dimension of the underlying space. Concentration inequalities and their applications have become an integral part of modern probability theory. See, for example, the seminal articles by Talagrand [43, 44, 45, 46, 47]. An excellent account can be found in the monograph [29] by Ledoux to which we refer the reader for a survey of the (pre-2001) literature.

Throughout this article our sample space is going to be some subset of the space of continuous function on $[0, \infty)$ denoted by $C[0, \infty)$ and products of such spaces. A sample path is denoted by $\omega(t)$, $0 \leq t < \infty$. The filtration will be the natural filtration made right-continuous and suitably completed under the probability measures we consider. The metric on the sample space will be mostly given by the uniform metric: $d(\omega, \omega') = \sup_t |\omega(t) - \omega'(t)|$.

The probability measures we consider on the above sample space are the laws of multidimensional semimartingales including diffusions. The reason for considering diffusion laws on the *path space* has some strong motivation from applications. For example, consider the boundary local time of a reflected Brownian motion in an infinite wedge. Despite its significance in several areas of probability including queueing theory and mathematical finance (see, for example, the excellent survey by Fernholz and Karatzas [17] for applications to Stochastic Portfolio Theory), very

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little is known about such local times. The challenge is the fact that such local times are functions of the entire history of the path (as opposed to being functions of the one dimensional marginals). We show how to estimate the fluctuation of such local times and explicitly compute the case of Brownian motion with constant drift reflected in the orthant $\{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \dots \geq x_n\}$. Two points are worth drawing attention to: reflected Brownian motion is a highly dependent system of processes; and, unlike a typical function concentration result, local times are not Lipschitz functions of the reflected process under the uniform norm. To the best of our knowledge there is no other way established in the literature to obtain concentration estimates of local times.

The other compelling reason comes from mathematical finance. Given a financial market with a large (typically hundreds or thousands) number of stocks, one hedges risk by constructing a *diversified portfolio*. Informally, this amounts to distributing the capital in holding and rebalancing shares over the entire equity market so that the value of the resulting portfolio is impervious to market risk. It is now clear that in mathematical terms this amounts to a concentration of the value process around a certain deterministic path. In Section 3.3 we show through examples how such conclusions might be reached from our results on measure concentration.

Before we proceed, let us add two caveats. One, we cannot compute expectations from concentration of measures, which require other methods. Two, because of the highly dependent structure in some of our examples our concentration bounds are not always Gaussian nor dimension independent. Gaussian tail estimate itself is a delicate property not shared by all stochastic processes. For example, consider the Bessel-square processes (see Revuz and Yor [40]). The marginal distribution of these diffusions are Chi-squares, which do not have sub-Gaussian tails. Thus, one cannot expect a Gaussian measure concentration property to hold for all Lipschitz functions of such processes.

Our proofs depend on an original observation due to K. Marton [30, 31, 32]: Concentration of measure is a consequence of what are known as *Transportation Cost Inequalities* (TCI). We explain this wonderful method in Section 2.1. In fact what we prove in the text is, under suitable assumptions, multidimensional diffusion measures satisfy Quadratic Transportation Cost Inequality (QTCI) (see also, Talagrand [47], Dembo [8], Dembo and Zeitouni [9]). Several other recent articles have taken a similar approach for proving concentration estimates for diffusions although ours is the first proof of QTCI for diffusions and other semimartingales w.r.t. the uniform metric. QTCI is unique in its advantages and is related to the log-Sobolev inequality, hypercontractivity, Poincaré inequality, inf-convolution, and Hamilton-Jacobi equations. For details, please consult Otto and Vilani [36], Bobkov and Götze [5], and Bobkov, Gentil, and Ledoux [4]. Also see the recent articles by Gozlan [20, 21] and by Gozlan et al. [22] which shows equivalence of QTCI with a restricted logarithmic Sobolev inequalities. The last article also gives the first proof that QTCI is preserved under bounded perturbation.

Let us provide a brief review of literature of measure concentration in path space. Houdré and Privault [24] and Nourdin and Viens [35] use tools from Malliavin Calculus to derive concentration inequalities for functionals on the Wiener space among other things. Proving TCI on the Wiener space using Girsanov theorem, as we have done, first appeared in Feyel and Ustunel [18]. Djellout, Guillin, and Wu [10] provide characterization for \mathbf{L}^1 -TCI for diffusions. They also prove

QTCI for diffusions with respect to the Cameron-Martin \mathbf{L}^2 -metric. Several articles in analysis and geometry are also devoted to this topic. Fang and Shao [14, 15] consider TCI on abstract path spaces on connected Lie groups, Fang, Wang, and Wu [16] consider TCI w.r.t. the uniform metric for special diffusions on a complete Riemannian manifold; Gourcy and Wu [19] considers log-Sobolev inequalities under the \mathbf{L}^2 -metric; Wang [50, 51], studies generalized TCI on complete connected Riemannian manifolds; and Wu and Zhang [52] prove QTCI for the uniform metric under an \mathbf{L}^2 -contraction property of the diffusion semigroup.

The outline of the article is as follows. In the following section we explain the connection between TCI and measure concentration. In Section 3 we prove QTCI for semimartingale strong solutions of SDE's under suitable assumptions in one dimension. By using tensorization properties of QTCI, in Theorem 6 we extend concentration results to multidimensional processes with independent coordinates. However, independent coordinates are of limited appeal in applications. In Theorem 7 we use a perturbation argument to prove measure concentration properties for dependent processes that are locally absolutely continuous with respect to some multidimensional process with independent coordinates. The strength of the concentration depends on a Birnbaum-Orlicz norm of the Radon-Nikodým derivative of one measure with respect to the other. In Section 3.3 we discuss concentration of Lipschitz functions of diffusions. This includes concentration of regular and stochastic integrals with respect to diffusions.

Our main example is worked out in Section 4 where we work out concentration estimates of local times for Brownian motion interacting through their ranks. These non-trivial processes can be described as follows. Let $\delta_1, \delta_2, \dots, \delta_n$ be n real constants. Consider the following system of stochastic differential equations:

$$(1) \quad dX_i(t) = \sum_{j=1}^n \delta_j 1(X_i(t) = X_{(j)}(t)) dt + dW_i(t), \quad i = 1, 2, \dots, n.$$

Here $X_{(1)}(t) \geq X_{(2)}(t) \geq \dots \geq X_{(n)}(t)$ are the coordinates of the process in the decreasing order, and $W = (W_1, W_2, \dots, W_n)$ is an n -dimensional Brownian motion. The SDE models the movement of n particles as interacting Brownian motions such that at every time point, if we order the positions of the particles, then the i th ranked particle from the top gets a drift δ_i for $i = 1, 2, \dots, n$. As time evolves, the Brownian motions switch ranks and drifts, and hence their motion is determined by such time dependent interactions. Such processes have been considered in several recent articles. Among the more recent ones, see Banner, Fernholz, and Karatzas [2], Banner and Ghomrasni [5], McKean and Shepp [33], Pal and Pitman [37], Jourdain and Malrieu [27], Chatterjee and Pal [6, 7], Ichiba and Karatzas [25], Ichiba et al. [26], and Shkolnikov [42]. We refer the reader to the above articles for the list of applications of such models. They are similar in the discrete setting to the dynamic models of spin glasses studied by Arguin and Aizenman [1], Ruzmaikina and Aizenman [39], Shkolnikov [41].

As an example of a typical result one can derive from this theory, let us state a theorem we prove in Section 4.

Theorem 1. *Consider the model described in (1). Let $L_{j,j+1}(T)$ denote the local time at zero for the semimartingale $(X_{(j)} - X_{(j+1)})$ up to time T . For any choice of parameters $(\delta_1, \dots, \delta_n)$ and constant initial points for the coordinate processes the*

random variable $\chi = \max_{1 \leq j \leq n-1} L_{j,j+1}(T)$ satisfies the following tail estimate:

$$P\left(|\chi - m_\chi| \geq rn^{5/2}\right) \leq 2 \exp\left\{-\frac{r^2}{CT}\right\}, \quad r \geq 2\sqrt{2 \log 2}.$$

Here m_χ is the median of χ and C refers to a universal constant.

An improvement on this bound and other applications related to rank-based models have been done in the follow-up article by Pal and Shkolnikov [38]. Some of our results about TCI for multidimensional diffusions have been recently generalized in an article by Üstünel [49].

2. PRELIMINARIES

2.1. Transportation cost and concentration. Suppose (\mathcal{X}, d) is a complete separable metric space equipped with the Borel sigma-algebra. For all probability measures P and Q on this probability space, consider the p -th Wasserstein distance

$$\mathcal{W}_p(P, Q) = \inf_{\pi} [Ed(X, X')^p]^{1/p},$$

where the infimum is over all couplings of a pair of random elements (X, X') such that the marginal law of X is P and that of X' is Q .

Now, we fix a particular probability P . Suppose there is a constant $C > 0$ such that for all probability measures $Q \ll P$ we have

$$(2) \quad \mathcal{W}_p(P, Q) \leq \sqrt{2CH(Q|P)}.$$

Here H refers to the relative entropy $H(Q|P) = E^Q \log(dQ/dP)$. Then we say that P satisfies the \mathbf{L}^p Transportation Cost Inequality with the constant C . When $p = 2$, this is often called a Quadratic Transportation Cost Inequality (QTCI).

A function $f : \mathcal{X} \rightarrow \mathbb{R}$ will be called Lipschitz if there is a positive constant α for which

$$|f(x) - f(y)| \leq \alpha d(x, y), \quad x, y \in \mathcal{X}.$$

The constant α is then referred to as the Lipschitz constant. We shall call a function to be 1-Lipschitz if α can be taken to be one. Let \mathcal{L} denote the set of all 1-Lipschitz functions on (\mathcal{X}, d) . The (very short) proof of the following theorem can be found in Ledoux [29, p. 118] and the original article by Marton [30].

Theorem 2. *Suppose that P satisfies QTCI with constant C . Then, one has the following concentration estimate for all $r \geq 2\sqrt{2C \log 2}$:*

(i) *For any measurable set A such that $\mu(A) \geq 1/2$, one gets*

$$(3) \quad \mu(A_r) \geq 1 - \exp\{-r^2/8C\}, \quad A_r = \{x \in \mathcal{X} : d(x, A) \leq r\}.$$

(ii) *And for any $f \in \mathcal{L}$, one has*

$$(4) \quad P(x : |f(x) - m_f| \geq r) \leq 2e^{-r^2/8C},$$

where m_f is the median of f .

In general, any Wasserstein metric can be used to show (2), however the choice of $p = 2$ is important due to its tensorization property we describe below. For the proof see Ledoux [29, p. 122-123] (although our statement appears slightly different from Ledoux's monograph, they can be easily seen to be equivalent).

Theorem 3. Suppose $(\mathcal{X}_i, d_i, P_i)$, $i = 1, 2, \dots, n$ be n polish spaces with corresponding distances and probability measures on them. Consider the product metric space $\mathcal{X}^n = \mathcal{X}_1 \times \dots \times \mathcal{X}_n$ with the distance

$$\bar{d}(x, y) = \sqrt{\frac{1}{n} \sum_{i=1}^n d_i^2(x_i, y_i)}, \quad x, y \in \mathcal{X}^n,$$

and the product probability measure on it $P = \otimes P_i$. Suppose that every P_i satisfies the QTCI with the same constant C , then P satisfies QTCI with constant C/n .

Finally, we need the following lemma which will be useful in the later text. Its (short) proof can be found in [10].

Lemma 4. [Lemma 2.1 in [10]] Suppose μ is a measure on a metric space (E, d_E) that satisfies TCI with respect to the \mathcal{W}_p norm with a constant C . Let (F, d_F) be another metric space. If the map $\Psi : (E, d_E) \rightarrow (F, d_F)$ is Lipschitz, i.e.,

$$d_F(\Psi(x), \Psi(y)) \leq \alpha d_E(x, y), \quad \text{for all } x, y \in E,$$

then $\tilde{\mu} = \mu \circ \Psi^{-1}$ satisfies TCI with the \mathcal{W}_p norm with a constant $C\alpha^2$ on (F, d_F) .

3. CONCENTRATION OF DIFFUSION LAWS

Fix a finite positive time horizon T . Consider the metric space given by $\mathcal{X} = C[0, T]$, along with the norm

$$(5) \quad d(\omega, \omega') = \|\omega - \omega'\|_\infty(T) = \sup_{0 \leq t \leq T} |\omega(t) - \omega'(t)|, \quad \omega \in C[0, T].$$

Also consider the product space $\mathcal{X}^n = C^n[0, T]$. Let $\omega = (\omega_1, \dots, \omega_n)$ and $\omega' = (\omega'_1, \dots, \omega'_n)$ be two elements in \mathcal{X}^n . Then

$$(6) \quad \bar{d}(\omega, \omega') = \sqrt{\frac{1}{n} \sum_{i=1}^n \sup_{0 \leq t \leq T} |\omega_i(t) - \omega'_i(t)|^2} = \sqrt{\frac{1}{n} \sum_{i=1}^n \|\omega_i - \omega'_i\|_\infty^2(T)}.$$

3.1. Concentration for a fixed time horizon. Consider a progressively measurable drift function $b(s, \omega)$ from $[0, \infty) \times C[0, \infty)$ into \mathbb{R} . That is to say, $b(s, \cdot)$ depends on the entire history of the process until time s . In particular, the processes we consider below need not be Markov. However, we do assume that the diffusion parameter $\sigma(t, X)$ depends only t and $X(t)$. That is, σ is a function from $[0, \infty) \times \mathbb{R}$ into \mathbb{R} . The following result is under the above set-up.

Theorem 5. Fix a time point $T > 0$. Suppose X is a strong solution to the equation

$$(7) \quad dX(t) = b(t, X)dt + \sigma(t, X(t))dW(t), \quad 0 \leq t \leq T, \quad X(0) = x(0),$$

where the coefficients satisfy the global Lipschitz conditions

$$(8) \quad \begin{aligned} |b(t, \omega) - b(t, \omega')| &\leq K_1 \sup_{0 \leq s \leq t} |\omega(s) - \omega'(s)| = K_1 \|\omega - \omega'\|_\infty(t) \\ \text{and } |\sigma(t, x) - \sigma(t, y)| &\leq K_2 |x - y|, \quad 0 \leq t \leq T, \quad x, y \in \mathbb{R}. \end{aligned}$$

Additionally assume that diffusion coefficient is bounded, i.e., $0 \leq \sigma(t, x) \leq \kappa$. Here K_1, K_2, κ are positive constants.

Let P denote the law of X considered as a probability measure on the metric space $(C[0, T], \|\cdot\|_\infty)$. Then P satisfies the quadratic transportation cost inequality (2) for the choice of

$$(9) \quad C = 4\kappa^2 T e^{4T(K_1^2 T + 4K_2^2)}.$$

Proof. The set-up of the proof is very similar to the proof of Theorem 5.6 in [10]. So we can conveniently skip some details. Consider the canonical sample space $\Omega = C[0, T]$ with a *universal Brownian filtration* $\{\mathcal{F}_t, 0 \leq t \leq T\}$. The details of this construction can be found in Revuz and Yor [40, Sec. 2 Chap. 3]. This amounts to a filtration that is generated by the coordinate process and suitably made right-continuous and *augmented* with the null sets that are common to the family of Wiener measures starting from any initial probability distribution.

We consider the standard Wiener measure on this space, and thus the coordinate process, $W(t)$, $0 \leq t \leq T$, is distributed as a standard Brownian motion. Since X is a strong solution to the SDE (7), one can construct a copy of the process X adapted to the above probability space that satisfies (7) with respect to the coordinate Brownian motion W . Let P denote the law of the process X on $C[0, T]$.

Let Q be any probability measure absolutely continuous with respect to P . Thus, if we define $M(T)$ to be the Radon-Nikodým derivative of Q with respect to P it is a function of X . In particular, $M(T)$ is a measurable function with respect to the Brownian filtration constructed above. We now construct a martingale by defining

$$(10) \quad M(t) = E(M(T) \mid \mathcal{F}_t), \quad 0 \leq t \leq T.$$

Then $M(t)$ is a martingale with respect to the Brownian filtration. It follows then, see Revuz and Yor [40, Sec. 3 Chap. 5], that this martingale has a continuous version which can be written as a stochastic integral

$$(11) \quad M(t) = 1 + \int_0^t H(s) dW(s), \quad 0 \leq t \leq T,$$

for some predictable process H .

It suffices for QTCI to consider Q such that the Radon-Nikodým derivative is square-integrable under P . We will assume that the martingale M is square-integrable for simplicity.

Now, since $M(t)$ is a continuous martingale, we use Girsanov's Theorem [40, p. 327]. Thus, under the measure Q , the process X satisfies the SDE

$$(12) \quad \begin{aligned} dX(t) &= b(t, X)dt + \sigma(t, X(t)) [dB(t) + M(t)^{-1}H(t)dt] \\ &= [b(t, X) + \sigma(t, X(t))M(t)^{-1}H(t)] dt + \sigma(t, X(t))dB(t). \end{aligned}$$

Here $B(t)$ is a Brownian motion under Q . Note that M is never zero under Q and hence M^{-1} makes perfect sense.

Now to bound the Wasserstein distance between the two measures P and Q , we need to couple the solutions of the two SDE's (7) and (12). To do this, we construct a solution of (12) on a filtered probability space with the Brownian motion $B(t)$ running during time $[0, T]$, and use this same Brownian motion to create a strong solution of (7).

Thus we have a coupled process:

$$\begin{aligned} X^{(1)}(t) &= x_0 + \int_0^t \left[b(s, X^{(1)}) + \sigma(s, X^{(1)}(s))M(s)^{-1}H(s) \right] ds + \int_0^t \sigma(s, X^{(1)}(s))dB(s), \\ X^{(2)}(t) &= x_0 + \int_0^t b(s, X^{(2)})ds + \int_0^t \sigma(s, X^{(2)}(s))dB(s). \end{aligned}$$

We now estimate the uniform distance between these two processes.

Define a nondecreasing sequence of functions $\psi_n(x)$ which serve as a smooth approximation to the function $|x|$ while satisfying

$$(13) \quad |\psi'_n(x)| \leq 1, \quad \text{and} \quad 0 \leq \psi''_n(x) \leq \frac{2}{nK_2x^2}.$$

The details on how such a sequence can be constructed can be found in [28, p. 291].

Now define

$$\begin{aligned} \Delta(t) := X^{(1)}(t) - X^{(2)}(t) &= \int_0^t \left[b(s, X^{(1)}) - b(s, X^{(2)}) + \sigma(s, X^{(1)}(s))M(s)^{-1}H(s) \right] ds \\ &\quad + \int_0^t \left[\sigma(s, X^{(1)}(s)) - \sigma(s, X^{(2)}(s)) \right] dB(s). \end{aligned}$$

Then by Itô's rule

$$\begin{aligned} \psi_n(\Delta(t)) &= \int_0^t \psi'_n(\Delta(s)) \left[b(s, X^{(1)}) - b(s, X^{(2)}) \right] ds \\ &\quad + \int_0^t \psi'_n(\Delta(s)) \sigma(s, X^{(1)}(s))M(s)^{-1}H(s) ds \\ (14) \quad &\quad + \frac{1}{2} \int_0^t \psi''_n(\Delta(s)) \left[\sigma(s, X^{(1)}(s)) - \sigma(s, X^{(2)}(s)) \right]^2 ds \\ &\quad + \int_0^t \psi'_n(\Delta(s)) \left[\sigma(s, X^{(1)}(s)) - \sigma(s, X^{(2)}(s)) \right] dB(s). \end{aligned}$$

Now, by the condition (8) on σ and the property of the function ψ''_n in (13) we get

$$(15) \quad 0 \leq \frac{1}{2} \int_0^t \psi''_n(\Delta(s)) \left[\sigma(s, X^{(1)}(s)) - \sigma(s, X^{(2)}(s)) \right]^2 ds \leq \frac{t}{n}.$$

By using the Lipschitz property of the drift function b and (13) we obtain

$$(16) \quad \left| \int_0^t \psi'_n(\Delta(s)) \left[b(s, X^{(1)}(s)) - b(s, X^{(2)}(s)) \right] ds \right| \leq K_1 \int_0^t \sup_{0 \leq u \leq s} |\Delta(u)| ds.$$

Let $\xi(s)$ be the process $M(s)^{-1}H(s)$. Since σ and ψ'_n are bounded functions, we get

$$(17) \quad \left| \int_0^t \psi'_n(\Delta(s)) \sigma(s, X^{(1)}(s))M(s)^{-1}H(s) ds \right| \leq \kappa \int_0^t |\xi(s)| ds.$$

The final stochastic integral above in (14) is a local martingale. We use Doob's \mathbf{L}^2 inequality [40, p. 54] to claim the following:

(18)

$$\begin{aligned} E \sup_{0 \leq s \leq t} \left| \int_0^s \psi'_n(\Delta(u)) \left[\sigma(u, X(u)^{(1)}) - \sigma(u, X^{(2)}(u)) \right] dB(u) \right|^2 \\ \leq 4E \int_0^t (\psi'_n(\Delta(s)))^2 \left[\sigma(s, X(s)^{(1)}) - \sigma(s, X^{(2)}(s)) \right]^2 ds \leq 4K_2^2 E \int_0^t \Delta^2(s) ds. \end{aligned}$$

The final inequality is due to the Lipschitz property of the coefficient σ and (13).

Combining the inequalities (14), (15), (16), (17), (18), and applying Cauchy-Schwarz we get

$$\begin{aligned} \frac{1}{4} E \sup_{0 \leq s \leq t} \psi_n^2(\Delta(s)) &\leq K_1^2 E \left(\int_0^t \sup_{0 \leq u \leq s} |\Delta(u)| ds \right)^2 \\ &\quad + \kappa^2 E \left(\int_0^t |\xi(s)| ds \right)^2 + \frac{t^2}{n^2} + 4K_2^2 E \int_0^t \Delta^2(s) ds \\ (19) \quad &\leq K_1^2 t \int_0^t E \sup_{0 \leq u \leq s} |\Delta(u)|^2 ds + \kappa^2 E \left(\int_0^t |\xi(s)| ds \right)^2 \\ &\quad + \frac{t^2}{n^2} + 4K_2^2 \int_0^t E \sup_{0 \leq u \leq s} |\Delta(u)|^2 ds \\ &\leq a_n + (K_1^2 T + 4K_2^2) \int_0^t E \sup_{0 \leq u \leq s} |\Delta(u)|^2 ds, \end{aligned}$$

where the constant a_n is given by (again applying Cauchy-Schwarz)

$$a_n = \kappa^2 T E \int_0^T \xi^2(s) ds + T^2/n^2.$$

Now, one can construct ψ_n such that $\psi_n(x)$ converges to $|x|$ uniformly on compact sets. Recalling that the sample paths are continuous and taking the limit as n goes to infinity in (19), we obtain

$$E \sup_{0 \leq s \leq t} \Delta^2(t) \leq 4a + 4(K_1^2 T + 4K_2^2) \int_0^t E \sup_{0 \leq u \leq s} |\Delta(u)|^2 ds, \quad a = \kappa^2 T E \int_0^T \xi^2(s) ds.$$

Let $\phi(t)$ denote the quantity $E \sup_{0 \leq s \leq t} \Delta^2(t)$. Then it follows from above that

$$\phi(t) \leq 4a + 4(K_1^2 T + 4K_2^2) \int_0^t \phi(s) ds.$$

By an application of Gronwall's lemma [40, Sec. 1, Appendix], we obtain $\phi(t) \leq 4ae^{4(K_1^2 T + 4K_2^2)t}$. Thus we get

$$E \sup_{0 \leq t \leq T} \Delta(t)^2 \leq 4\kappa^2 T e^{4T(K_1^2 T + 4K_2^2)} E \int_0^T \xi^2(s) ds.$$

Recall that we are using the uniform distance $d(\omega, \omega') = \sup_{0 \leq t \leq T} |\omega(t) - \omega'(t)|$ between two paths ω, ω' in $C[0, T]$. Thus, from our calculations above, we get

$$(20) \quad E d^2(X^{(1)}, X^{(2)}) \leq 4\kappa^2 T e^{4(K_1^2 T + 4K_2^2)T} E \int_0^T \xi^2(s) ds.$$

On the other hand (see [10, eqn. 5.7]), the entropy of Q with respect P is given by

$$(21) \quad H(Q | P) = \frac{1}{2} E^Q \int_0^T \xi(u)^2 du.$$

By combining the above inequality with (20) we get

$$Ed^2(X^{(1)}, X^{(2)}) \leq 8\kappa^2 T e^{4T(K_1^2 T + 4K_2^2)} H(Q | P),$$

which completes the proof.

The final claim follows by the tensorization argument outlined in the previous section. \square

Theorem 6. *For each $i = 1, 2, \dots, n$, let $b_i(t, \omega)$ be progressively measurable real-valued drift function on $[0, \infty) \times C[0, \infty)$ and $\sigma_i(t, x)$ be measurable functions from $[0, \infty) \times \mathbb{R}$ into \mathbb{R} . Consider the following system of multidimensional stochastic differential equation:*

$$dX_i(t) = b_i(t, X_i)dt + \sigma_i(t, X_i(t))dW_i(t), \quad 0 \leq t \leq T, \quad X_i(0) = x_i.$$

Assume that there are positive constants K and κ such that for every $i = 1, 2, \dots, n$, the coefficients satisfy

$$(22) \quad \begin{aligned} & \max_{1 \leq i \leq n} |b_i(t, \omega) - b_i(t, \omega')| \leq K \|\omega - \omega'\|_\infty(t), \quad \omega, \omega' \in C[0, \infty) \\ & \max_{1 \leq i \leq n, 0 \leq t \leq T} |\sigma_i(t, x) - \sigma_i(t, y)| \leq K |x - y|, \quad x, y \in \mathbb{R} \\ & \text{and} \quad 0 \leq \min_{1 \leq i \leq n} \sigma_i(t, x) \leq \max_{1 \leq i \leq n} \sigma_i(t, x) \leq \kappa. \end{aligned}$$

Let P denote the product law of independent processes (X_1, \dots, X_n) . Then P , seen as a probability measure on the metric space $(C^n[0, 1], \bar{d})$, as in (6), satisfies the QTCI (2) for the choice of $C = 4n^{-1}\kappa^2 T e^{K^2 T(T+4)}$.

In applications however the multidimensional diffusions with independent coordinates are of limited use, although it is common to use dependent diffusions that are a Girsanov change of measure of independent ones. If this change of measure is not too drastic, one should expect concentration properties to transfer to the dependent diffusion case. The following perturbation result makes this precise.

We prove the result for a general metric space and any TCI. However, for our purpose in this paper the metric space will be the path space and the metric will be the uniform metric.

Theorem 7. *Suppose P and R are mutually absolutely continuous probability measures on a complete separable metric space (\mathcal{X}, d) . Let L be the logarithm of the Radon-Nikodým derivative process of R with respect to P . Suppose that P satisfies \mathbf{L}^p TCI ($p \geq 1$) with a constant C . Consider any A such that $R(A) \geq 1/2$. For all r such that*

$$r \geq 2\sqrt{2C \log 2 + 4C \|L\|_1},$$

one has

$$(23) \quad 1 - R(A_r) \leq \exp\left(-\frac{r^2}{8C(1 + 4\|L\|_\Phi)}\right).$$

Here, $\|L\|_1$ is the expectation w.r.t. R , and $\|L\|_\Phi$ is the Birnbaum-Orlicz norm (w.r.t. R) defined by

$$(24) \quad \|L\|_\Phi := \inf \{a > 0 : E^R \Phi(|L|/a) \leq 1\}, \quad \Phi(t) = e^t - t - 1.$$

Proof. Let A be a measurable subset and let A_r be as described in (3). For any measurable subset B , let ν_B denote the probability measure R , conditioned on B , i.e.,

$$\nu_B(\cdot) = \frac{R(B \cap \cdot)}{R(B)}.$$

The measure ν_B is clearly dominated by R , and hence by P , due to the assumed mutual absolute continuity.

Consider the Wasserstein distance between ν_A and ν_B , where B is the complement of A_r . By the triangle inequality and the fact that P satisfies TCI, we get

$$(25) \quad \mathcal{W}_p(\nu_A, \nu_B) \leq \mathcal{W}_p(\nu_A, P) + \mathcal{W}_p(\nu_B, P) \leq \sqrt{2CH(\nu_A | P)} + \sqrt{2CH(\nu_B | P)}.$$

Now, since $d(A, B) \geq r$, for any coupling between $X \sim \nu_A$ and $Y \sim \nu_B$, it follows immediately that $d(X, Y) \geq r$. Thus, it follows that the left side of the above inequality is at least as large as r . We estimate the right side below.

Obviously

$$H(\nu_A | P) = E^{\nu_A} \left(\log \frac{d\nu_A}{dP} \right) = E^{\nu_A} \left(\log \frac{d\nu_A}{dR} \right) + E^{\nu_A} \left(\log \frac{dR}{dP} \right).$$

In other words

$$(26) \quad H(\nu_A | P) \leq H(\nu_A | R) + \max \left(0, E^{\nu_A} \left(\log \frac{dR}{dP} \right) \right).$$

Now, by assumption, $\log dR/dP = L$. Thus

$$(27) \quad E^{\nu_A} \left(\log \frac{dR}{dP} \right) = \frac{1}{R(A)} E^R(1\{A\}L) \leq \frac{1}{R(A)} E^R(1\{A\}L).$$

Select A such that $R(A) \geq 1/2$. Then, we get

$$(28) \quad \frac{1}{R(A)} E^R(1\{A\}L) \leq (R(A))^{-1} E^R L = 2 \|L\|_1.$$

For the set B , we follow a similar line of argument except for the final estimate above. We use a pair of Young's function (i.e., convex conjugates, see Neveu [34, Appendix, p. 210-213]) Φ and Ψ given by

$$\Phi(t) = e^t - t - 1, \quad \Psi(v) = (1+v) \log(1+v) - v.$$

Recall the definition of a Birnbaum-Orlicz norm for suitable random variables:

$$\|X\|_\Phi := \inf \{a > 0 : E^R \Phi(|X|/a) \leq 1\}, \quad \|Y\|_\Psi := \inf \{a > 0 : E^R \Psi(|Y|/a) \leq 1\}.$$

We will use the following generalization of the Hölder's inequality for Young functions (see [34, Appendix, p. 210-213]):

$$\frac{1}{R(B)} E^R(1\{B\}L) \leq 2 \|L\|_\Phi \|1\{B\}/R(B)\|_\Psi.$$

Let us estimate $\|1\{B\}/R(B)\|_\Psi$. For any $a > 0$, we get

$$(29) \quad \begin{aligned} E^R \Psi \left[\frac{1\{B\}}{aR(B)} \right] &= E^R \left\{ \left(1 + \frac{1\{B\}}{aR(B)} \right) \log \left(1 + \frac{1\{B\}}{aR(B)} \right) - \frac{1\{B\}}{aR(B)} \right\} \\ &= R(B) \left(1 + \frac{1}{aR(B)} \right) \log \left(1 + \frac{1}{aR(B)} \right) - \frac{1}{a}. \end{aligned}$$

Note that $R(B) < 1/2$ since $R(A) \geq 1/2$. We claim that

$$(30) \quad \|1\{B\}/R(B)\|_\Psi \leq a := \frac{\log \frac{1}{R(B)}}{1 - R(B) \log \frac{1}{R(B)}}.$$

It suffices to check that for this value of a , the expression in (29) is smaller than one. To see this, note that, by our definition

$$1 + \frac{1}{aR(B)} = \frac{1}{R(B) \log 1/R(B)}.$$

Thus

$$\begin{aligned} E^R \Psi \left[\frac{1\{B\}}{aR(B)} \right] &= R(B) \left(1 + \frac{1}{aR(B)} \right) \log \left(1 + \frac{1}{aR(B)} \right) - \frac{1}{a} \\ &= \frac{1}{\log(1/R(B))} \log \frac{1}{R(B) \log 1/R(B)} - \frac{1}{\log 1/R(B)} + R(B) \\ &= 1 + \frac{1}{\log(1/R(B))} \log \frac{1}{\log 1/R(B)} - \frac{1}{\log 1/R(B)} + R(B) \end{aligned}$$

Let $1/x := -\log R(B)$. Our claim will follow if we show the following to be negative:

$$h(x) = x \log x - x + e^{-1/x} = \frac{1}{\log(1/R(B))} \log \frac{1}{\log 1/R(B)} - \frac{1}{\log 1/R(B)} + R(B).$$

Since $0 \leq R(B) \leq 1/2$ it is enough to check in the interval $(0, 1/\log 2)$. We claim that in this interval the function h is convex. To verify, note that

$$h'(x) = \log x + x^{-2}e^{-1/x}, \quad h''(x) = \frac{1}{x} - \frac{2}{x^3}e^{-1/x} + \frac{1}{x^4}e^{-1/x}.$$

Noting that $e^{1/x} \geq 1 + 1/x$, we get

$$\begin{aligned} h''(x) &= x^{-4}e^{-1/x} [x^3e^{1/x} - 2x + 1] \geq x^{-4}e^{-1/x} [x^3 + x^2 - 2x + 1] \\ &= x^{-4}e^{-1/x} [x^3 + (x-1)^2] \geq 0. \end{aligned}$$

This shows that h is convex. Thus, to show check for the negative sign of h it is enough to check at the end points. Plainly $h(0+) = 0$, and numerically $h(1/\log 2) \approx -0.41 < 0$. By convexity it now follows that that $h(x)$ is negative for $x \in (0, 1/\log 2)$.

This proves the claim (30). In fact, we will simplify our choice of a slightly more by defining

$$a = 2 \log \frac{1}{R(B)},$$

which is larger than the choice in (30) since $R(B) \leq 1/2$.

Combining our argument so far we obtain

$$(31) \quad \frac{1}{R(B)} E^R (1\{B\}L) \leq 4 \|L\|_\Phi \log \frac{1}{R(B)}.$$

Thus, combining the above with (25), (26), (27), (28) we get

$$\frac{r}{\sqrt{2C}} \leq \sqrt{H(\nu_A | R) + 2\|L\|_1} + \sqrt{H(\nu_B | R) + 4\|L\|_\Phi \log \frac{1}{R(B)}}$$

Note that

$$H(\nu_A | R) = \log \frac{1}{R(A)} \leq \log 2, \quad H(\nu_B | R) = \log \frac{1}{R(B)}.$$

Thus

$$\frac{r}{\sqrt{2C}} \leq \sqrt{\log 2 + 2\|L\|_1} + \sqrt{(1 + 4\|L\|_\Phi) \log \frac{1}{R(B)}}.$$

Note that $R(B) = 1 - R(A_r)$. Hence for all r larger than

$$2\sqrt{2C \log 2 + 4C\|L\|_1},$$

one has (say)

$$2C(1 + 4\|L\|_\Phi) \log \frac{1}{1 - R(A_r)} \geq \frac{r^2}{4}.$$

Or, by rearranging terms, we get

$$1 - R(A_r) \leq \exp\left(-\frac{r^2}{8C(1 + 4\|L\|_\Phi)}\right).$$

This proves the assertion. \square

Remark. To get a feeling for the Birnbaum-Orlicz norm $\|\cdot\|_\Phi$ used in (24), let us compute this norm for the case when $L(T) = B(T)$, a one-dimensional standard Brownian motion. For any $a > 0$, a quick calculation will show

$$E\left(e^{a^{-1}|B(T)|}\right) - \frac{1}{a}E|B(T)| - 1 = 2e^{T/2a^2}\Phi\left(\sqrt{T}/a\right) - \frac{1}{a}\sqrt{\frac{2T}{\pi}} - 1.$$

If we take $a = \sqrt{T}$, the above expression reduces to $2\sqrt{e}\Phi(1) - \sqrt{2/\pi} - 1$ which comes to about 0.976. In other words $\|B(T)\|_\Phi \approx \sqrt{T}$.

Finally note that so far we have assumed that the starting points of the processes are given constants. When we randomize the starting values, it is not obvious what happens to the Transportation Cost Inequality of the mixture. This general problem is studied under the rubric of dependent tensorization and TCI (specialized to Markov chains of size two). See [30] and [10] for more details, in particular Marton's coupling for Markov chains.

However, if we are interested in only concentration of measures, certain bounds can be easily obtained.

Lemma 8. *Suppose μ is a probability measure on a metric space (E, d_E) . Let (F, d_F) be another metric space. Suppose there is a regular conditional probability P_x , $x \in E$, which is a probability measure on the Borel σ -algebra of F . Assume that each P_x in the support of μ satisfies QTCI with a constant C .*

For any Lipschitz function $f : F \rightarrow \mathbb{R}$, let $m_f(x)$ denote the median of f with respect to the probability measure P_x . Let m denote any constant. Then, if P denotes the randomized measure $\int P_x(\cdot)\mu(dx)$, one gets

$$P(|F - m| > r) \leq 2 \exp\left\{-\frac{r^2}{32C}\right\} + \mu(x : |m_f(x) - m| > r/2), \quad r \geq 2\sqrt{2 \log 2}.$$

Proof. Follows from the triangle inequality. \square

To specialize the above result to our set-up at hand, take (E, d_E) to be \mathbb{R}^n under the Euclidean norm and (F, d_F) to be $C^n[0, T]$ under the d norm. One can take m to be either the expectation or the median of the numbers $m_f(x)$. Much better bounds can be obtained if we know that the Markov semigroup of the diffusion has some contraction properties. See the analysis by Wu & Zhang [52] in this direction.

3.2. Concentration for infinite time horizon. For any two paths ω_1 and ω_2 in $C[0, \infty)$, we denote the uniform metric on their restriction to $[0, n]$ by $d_n(\omega_1, \omega_2) = \sup_{0 \leq t \leq n} (|\omega_1(t) - \omega_2(t)|)$. Then consider the locally uniform metric

$$(32) \quad \rho(\omega_1, \omega_2) = \max_n \frac{c_n d_n(\omega_1, \omega_2)}{1 + d_n(\omega_1, \omega_2)}$$

for some sequence of positive numbers c_n (to be specified later) such that $\lim_{n \rightarrow \infty} c_n = 0$. It is well-known that this metric makes the space $C[0, \infty)$ a complete separable metric space.

The concentration results of the last subsection can all be extended to this case, although presumably it is less useful since it is more difficult to check Lipschitz-ness of functions with respect to the local metric. We include a statement for mathematical completeness specialized to arbitrary finite stopping times.

Consider a stopping time τ with respect to the right continuous filtration on $C[0, \infty)$ that is continuous with respect to metric ρ . Denote by $C[0, \tau]$ the metric space of paths in $C[0, \infty)$ such that $\tau(\omega) < \infty$ and $\omega(t) = \omega_\tau$ for all $t \geq \tau$. Clearly $C[0, \tau]$ is a closed subset of a Polish space, and is hence Polish itself under ρ .

Theorem 9. *Consider the same one-dimensional process as in Theorem 5 (with $K_1 = K_2 = K$) stopped at a continuous stopping time τ . Let P denote the law of the stopped process. Then P satisfies QTCI with the constant \tilde{C} given by*

$$\tilde{C} = 4\kappa^2 \max_n c_n^2 n e^{4K^2(n+4)}.$$

In particular, if we choose $c_n = n^{-1/2} \exp(-2K^2(n+4))$ in (32), then one can take $\tilde{C} = 4\kappa^2$.

Proof. The proof is very similar to the proof of Theorem 5. Consider the coupling of processes $X^{(1)}$ and $X^{(2)}$ using the same driving Brownian motion, and consider the stopped processes $X_{t \wedge \tau_1}^{(1)}$ and $X_{t \wedge \tau_2}^{(2)}$. Here τ_1 and τ_2 are copies of τ applied to paths of $X^{(1)}$ and $X^{(2)}$ respectively. Let $\Delta(t) = X_{t \wedge \tau_1}^{(1)} - X_{t \wedge \tau_2}^{(2)}$.

Notice that the argument in the proof of Theorem 5 goes through for stopping times until inequality (19) which gets now modified to

$$(33) \quad \frac{1}{4} E \sup_{0 \leq s \leq t} \psi_n^2(\Delta(s)) \leq a_n(t) + K^2(t+4) \int_0^t E \Delta(s)^2 ds.$$

Here

$$a_n(t) = \kappa^2 t E \int_0^{t \wedge \tau_1} \xi(s)^2 ds + \frac{t^2}{n^2}.$$

Thus, as before, by Gronwall's lemma we get

$$(34) \quad E d_n^2(X^{(1)}, X^{(2)}) = E \sup_{0 \leq s \leq n} \Delta(s)^2 \leq 4\kappa^2 n e^{4K^2(n+4)} E \int_0^{n \wedge \tau_1} \xi(s)^2 ds.$$

Now,

$$\begin{aligned}
E\rho^2(X^{(1)}, X^{(2)}) &\leq \max_n c_n^2 E \left[\frac{d_n^2(X^{(1)}, X^{(2)})}{1 + d_n^2(X^{(1)}, X^{(2)})} \right] \\
&\leq 4\kappa^2 \max_n c_n^2 n e^{4K^2(n+4)} E \int_0^{t \wedge \tau_1} \xi(s)^2 ds \\
&\leq \tilde{C} E \int_0^{\tau_1} \xi(s)^2 ds.
\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 5. \square

3.3. Classes of lipschitz functions. In this section our objective is to work out a list natural examples of functions on the path space that are Lipschitz with respect to the uniform norm. Our aim is to show that paths of random processes derived from multidimensional diffusions lie in a cylinder around its “expected path” with exponentially high probability. Toward that aim, under suitable assumptions, we show concentration of processes of the type $\int_0^t \pi(u) du$ where π is an adapted process, and of adapted local martingales.

We specialize to the case of $T = 1$ and 1-Lipschitz functions. Any other value of T or of Lipschitz constant can be reduced to this case by scaling space and time.

Lemma 10. *Suppose $\{f(t), 0 \leq t \leq 1\}$ be a collection of functions $f(t) : C^n[0, 1] \rightarrow \mathbb{R}$ which are 1-Lipschitz with respect to \bar{d} . That is, if ω and ω' are elements in $C^n[0, 1]$, then*

$$|f(t)(\omega) - f(t)(\omega')|^2 \leq \frac{1}{n} \sum_{i=1}^n \|\omega_i - \omega'_i\|_\infty^2, \quad \text{for every } 0 \leq t \leq 1.$$

Then the following functions are also 1-Lipschitz.

- (i) $\sup_t f(t)$ when the supremum is measurable.
- (ii) For any Lipschitz function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, the composition $\phi \circ f(t)$. In particular, $-f(t)$, $|f(t)|$, and $|f(t) - a(t)|$, where $a(t)$ is any non-random function.
- (iii) The functions $g(t) = \int_0^t f(u) du$, $0 \leq t \leq 1$.

Further, suppose $f(t)$ is not known to be a priori Lipschitz. Let $\omega = (\omega_1, \dots, \omega_n)$, where each $\omega_i \in C[0, 1]$. Let $f^i(t)$ denote the function $f(t)$ as a function of ω_i , while the rest of the coordinates are kept constant. Then, if for every choice of i and ω_j , $j \neq i$, the function $f^i(t)$ is Lipschitz in ω_i with coefficient $1/n$, i.e.,

$$|f^i(t)(\omega_i) - f^i(t)(\omega'_i)| \leq \frac{1}{n} \|\omega_i - \omega'_i\|_\infty,$$

then $f(t)$ is 1-Lipschitz with respect to the \bar{d} norm.

Proof. The proofs of (i), (ii), and (iii) are obvious. To see the second part, choose a pair ω, ω' in the product space. Construct a sequence of vectors $\eta(1), \dots, \eta(n+1)$ in $C^n[0, 1]$ such that

$$\eta_j(i) = \begin{cases} \omega'_j, & \text{if } j \leq i-1, \\ \omega_j, & \text{otherwise.} \end{cases}$$

Thus $\eta(1) = \omega$ and $\eta(n+1) = \omega'$.

By the triangle inequality and the property of being separately Lipschitz in each coordinate, we get

$$\begin{aligned} |f(t)(\omega) - f(t)(\omega')|^2 &\leq n \sum_{i=1}^n |f(t)(\eta(i+1)) - f(t)(\eta(i))|^2 \\ &\leq n \sum_{i=1}^n \frac{1}{n^2} \|\omega_i - \omega'_i\|_\infty^2 = \frac{1}{n} \sum_{i=1}^n \|\omega_i - \omega'_i\|_\infty^2. \end{aligned}$$

This shows that $f(t)$ is 1-Lipschitz with respect to the \bar{d} norm. \square

We have the following corollaries. We focus on the set-up in Theorem 6, although please keep in mind that for the following Gaussian concentration bounds and \mathbf{L}^p TCI would suffice. In particular they hold for diffusions satisfying the conditions in Corollary 4.1 in [10].

Corollary 11 (Concentration of regular integrals). *Consider the set-up in Theorem 6. Suppose that $\{f(t), 0 \leq t \leq 1\}$ is a measurable real-valued process on $C^n[0, 1]$ such that each $f(t)$ is 1-Lipschitz with respect to the metric \bar{d} . Consider the process of integrals*

$$g(t)(\omega) = \int_0^t f(u)(\omega) du, \quad \text{and} \quad \mu(t) = \text{median of } g(t), \quad 0 \leq t \leq 1.$$

Then, when (22) holds, we have

$$P \left(\left| \sup_{0 \leq t \leq 1} |g(t) - \mu(t)| - \bar{\mu} \right| \geq r \right) \leq 2e^{-nr^2/8\sigma^2},$$

for all $r \geq 2\sigma\sqrt{2n^{-1}\log 2}$ where $\bar{\mu}$ is the median of $\sup_{0 \leq t \leq 1} |g(t) - \mu(t)|$ under the measure P .

The proof is straightforward application of Lemma 10 conclusion (iii). The supremum is measurable since $g(t)$ is always continuous.

The last two corollaries are significant in mathematical finance where the value process of a portfolio is often expressed as a stochastic integral. In particular, it is a martingale under, what is known as, the *risk-neutral* measure. The final value of such a martingale is often determined externally (i.e., pay-off from an European derivative). These corollaries together with Theorem 7 can determine whether such portfolios can be impervious to random *market risk*.

Corollary 12 (Concentration of martingales). *Consider the set-up in Theorem 6 (iii). Let $\{N(t), 0 \leq t \leq 1\}$ be a P -martingale such that $N_0 = 0$ and N_1 is a Lipschitz function with respect to the metric \bar{d} . Then the following concentration inequality holds*

$$(35) \quad P \left(\left| \sup_{0 \leq t \leq 1} |N(t)| - \bar{\nu} \right| > r \right) \leq 2e^{-nr^2/8\sigma^2},$$

for all $r \geq 2\sigma\sqrt{2n^{-1}\log 2}$, where $\bar{\nu} = E^P \sup_{0 \leq t \leq 1} |N(t)|$.

Proof. This is straightforward since N_1 is Lipschitz implies every other $N(t)$, which are conditional expectations of N_1 , must also be Lipschitz. \square

3.4. Local times. Now we come to the discussion of a particularly important class of functions in the study of continuous stochastic processes: the local time. Consider a standard Brownian motion and let $L(t)$ denote its local time at zero. Then, by definition (see [40, p. 227])

$$L(t) = \lim_{\epsilon \rightarrow 0} \frac{1}{2\epsilon} \int_0^t 1\{|B(s)| < \epsilon\} ds.$$

This is clearly not a Lipschitz function of the paths of $B(t)$. However, it is well-known that Brownian local time has Gaussian tails. In fact, a famous theorem of Lévy states that $L(t)$ has the same law as $|B(t)|$. One way to prove this concentration is by considering the Tanaka decomposition [40, p. 239]:

$$|B(t)| = \int_0^t \text{sgn}(B(s)) dB(s) + L(t).$$

If β denotes the process $\int_0^t \text{sgn}(B(s)) dB(s)$, then Lévy (and later, Skorokhod) showed that $L(t) = -\inf_{0 \leq s \leq t} \beta(s) \wedge 0$ (see [40, p. 239]). Thus $L(t)$ is a 1-Lipschitz function of the paths of β . Since β is another standard Brownian motion, it imparts Gaussian concentration to the local time function.

This notion of obtaining local time as a Skorokhod map has been greatly generalized. For the rest of text we will focus on such generalizations. We refer the reader to the articles by Dupuis and Ramanan [12, 13] from which we borrow our description of the so-called Skorokhod Problem which we describe below.

Consider a closed set $G \subseteq \mathbb{R}^n$ and a set of unit vectors $d(x)$ for each point x on the boundary of G (say ∂G). Let $D^n[0, \infty)$ be the set of maps from $[0, \infty)$ to \mathbb{R}^n that are right continuous with left limits. For $\tilde{\eta} \in D^n[0, \infty)$ let $|\tilde{\eta}|(T)$ be the total variation of $\tilde{\eta}$ on $[0, T]$ with respect to the Euclidean norm.

Definition 1. Let $\psi \in D^n[0, \infty)$ with $\psi_0 \in G$ be given. Then $(\phi, \tilde{\eta})$ solves the Skorokhod Problem (SP) for ψ with respect to G and d if $\phi_0 = \psi_0$, and if for all $t \in [0, \infty)$

- (i) $\phi(t) = \psi(t) + \tilde{\eta}(t)$;
- (ii) $\phi(t) \in G$;
- (iii) $|\tilde{\eta}|(t) < \infty$;
- (iv) $|\tilde{\eta}|(t) = \int_0^t 1\{\phi(s) \in \partial G\} d|\tilde{\eta}|(s)$;
- (v) There exists measurable $\gamma : [0, \infty) \rightarrow \mathbb{R}^n$ such that $\gamma(s) \in d(\phi(s))$, the set of direction vectors at the point $\phi(s)$ ($|\tilde{\eta}|$ -almost surely), and

$$\tilde{\eta}(t) = \int_0^t \gamma(s) d|\tilde{\eta}|(s).$$

When, the solution of SP exists (and is unique) for a large enough subset of the path space, the map that takes ψ to ϕ is called the Skorokhod map. The Skorokhod map, following an original idea due to Skorokhod, is used to construct stochastic processes that are constrained to remain within G by reflecting them inwards at the boundary ∂G in the direction given by the vector field d .

A Skorokhod map is Lipschitz if there exists a positive constant K such that if $(\phi, \tilde{\eta})$ and $(\phi', \tilde{\eta}')$ are solutions to the SP for ψ and ψ' respectively, one has

$$(36) \quad \begin{aligned} \sup_{t \geq 0} \|\tilde{\eta}(t) - \tilde{\eta}'(t)\| &\leq K \sup_{t \geq 0} \|\psi(t) - \psi'(t)\|, \\ \sup_{t \geq 0} \|\phi(t) - \phi'(t)\| &\leq K \sup_{t \geq 0} \|\psi(t) - \psi'(t)\|. \end{aligned}$$

Here $\|\cdot\|$ is the regular Euclidean norm. The constant K is then called the Lipschitz constant of the Skorokhod map and ψ is called the driving noise. Note that the norm used in (36) is weaker than the \bar{d} norm we have been using so far.

Corollary 13. *Suppose the Skorokhod map is 1-Lipschitz. Then, if the noise process ψ is chosen randomly with a law satisfying Theorem 6, then constrained random path ϕ and the local time function η also satisfies QTCI with respect to the metric*

$$\hat{d}(\omega, \omega') = \sup_{0 \leq t \leq T} \sqrt{\frac{1}{n} \sum_{i=1}^n (\omega_i(t) - \omega'_i(t))^2}.$$

As a corollary we obtain that reflected Brownian motion (RBM) satisfies QTCI with respect to the above norm whenever the above Skorokhod map is Lipschitz.

Proof. This is a corollary of the fact the QTCI is preserved under Lipschitz maps. See Lemma 4. The fact that $\hat{d} \leq \bar{d}$ is straightforward.

To obtain RBM, one needs to take the driving noise as a typical path of multi-dimensional Brownian motion that satisfies QTCI by Theorem 6. \square

Note that the above result for RBM is not useful since we do not know the QTCI constant (which will depend on n). This is a rather non-trivial job and something that we work out in detail for a specific example in the following Section. However, several natural conditions that guarantee when the Skorokhod map is Lipschitz can be found in [23], [11, Thm. 2.2], [12, Thm. 3.2] and [13, Thm. 2.2]. We will the following result from Dupuis and Ramanan [13].

Theorem 14 (Theorem 2.2 in [13]). *Consider the Skorokhod problem on a polyhedral domain $G = \{x \in \mathbb{R}^n : \langle x, \eta_i \rangle \geq c_i\}$ for some vectors η_i and scalars c_i , where $i = 1, 2, \dots, n$. Suppose the vector of direction of constraints (or, reflection) is constant on each face and are given by the vectors d_i , $i = 1, 2, \dots, n$ which are linearly independent and satisfy $\langle d_i, \eta_i \rangle = 1$ for each i . Define the matrix*

$$(37) \quad Q = [q_{ij}] = \begin{cases} |\langle d_i, \eta_j \rangle|, & \text{if } i \neq j, \\ |1 - \langle d_i, \eta_i \rangle|, & \text{if } i = j. \end{cases}$$

If the spectral radius of Q satisfies $\sigma(Q) < 1$, then the Skorokhod map is Lipschitz.

4. APPLICATIONS TO RANK-BASED MODELS

Consider the model described in (1) in the Introduction. We call such a model as an n particle rank-based model. For finite n , with arbitrary initial values of $X_i(0)$ and arbitrary drifts δ_i , the existence and uniqueness in law of such an n particle model is guaranteed by a standard application of Girsanov's Theorem. Please see Lemma 6 in [37]. A part of that lemma is reproduced below.

Let $\delta = (\delta_i, 1 \leq i \leq N) \in \mathbb{R}^N$ and let μ be an arbitrary probability distribution on \mathbb{R}^N . Consider the canonical sample space and filtration for the multidimensional Brownian motion described in Section 3. Let (X_1, \dots, X_n) denote the coordinate map. Let $P^{\delta, \mu}$ denote the law of the n -particle rank-based model where the initial position is distributed as μ . Thus, $P^{0, \mu}$ is the Wiener measure starting from μ .

Lemma 15. *For each $t > 0$ the law $P^{\delta, \mu}$ is absolutely continuous with respect to $P^{0, \mu}$ on \mathcal{F}_t , with density*

$$(38) \quad \exp \left(\sum_{j=1}^N \delta_j \beta_j(t) - \frac{t}{2} \sum_{j=1}^N \delta_j^2 \right)$$

where β_j can be defined by the expression

$$(39) \quad \beta_j(t) = \sum_{i=1}^N \int_0^t 1 \{ X_i(s) = X_{(j)}(s) \} dX_i(s), \quad 1 \leq j \leq N.$$

Under $P^{\delta, \mu}$ the β_j 's are independent Brownian motions on \mathbb{R} with drift coefficients δ_j and diffusion coefficient 1.

The other lemma we require considers the law of the ordered particle system under $P^{\delta, \mu}$:

$$X_{(1)}(t) \geq X_{(2)}(t) \geq \dots \geq X_{(n)}(t).$$

This ordered system, as shown in [2] and [37], is a reflected Brownian motion in the cone $\{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \dots \geq x_n\}$. The following is in Lemma 4 of [37], except that our ordering in (1) is the reverse of the notation used in [37].

Lemma 16. *Let $X_i, 1 \leq i \leq n$ be a solution of the SDE (1), defined on the canonical space, for some arbitrary initial condition and arbitrary drifts $\{\delta_i\}$. Then for each $1 \leq j \leq n$ the j th ordered process $X_{(j)}$ is a continuous semimartingale with decomposition*

$$(40) \quad dX_{(j)}(t) = d\beta_j(t) + \frac{1}{\sqrt{2}}(dL_{j,j+1}(t) - dL_{j-1,j}(t))$$

where the β_j 's for $1 \leq j \leq n$ are independent Brownian motions (with respect to the given filtration) with unit variance coefficient and drift coefficients δ_j , and are the same as appearing in (39).

Moreover, $L_{0,1} = L_{n,n+1} = 0$, and for $1 \leq j \leq n-1$

$$(41) \quad L_{j,j+1}(t) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1((X_{(j)}(s) - X_{(j+1)}(s))/\sqrt{2} \leq \epsilon) ds, \quad t \geq 0,$$

which is half the continuous increasing local time process at 0 of the semimartingale $(X_{(j+1)} - X_{(j)})/\sqrt{2}$. Moreover, the ordered system is a Brownian motion in the domain

$$(42) \quad \{x \in \mathbb{R}^n : x_1 \geq x_2 \geq \dots \geq x_n\}$$

with constant drift vector $(\delta_j, 1 \leq j \leq N)$ and normal reflection at each of the $n-1$ boundary hyperplanes $\{x_{(i)} = x_{(i+1)}\}$ for $1 \leq i \leq n-1$.

4.1. Concentration of intersection local times. Our objective is to show that the vector of boundary local times $(L_{j,j+1}, j = 1, 2, \dots, n-1)$ satisfy the QTCI. We start with a lemma.

Lemma 17. *Consider the rank-based model (1). Let \bar{X} denote the center of mass process*

$$\bar{X}(t) = \frac{1}{n} \sum_{i=1}^n X_i(t), \quad t \geq 0.$$

Then

- (i) $\bar{X}(t) - \bar{X}(0)$ is a Brownian motion with mean $\bar{\delta} = n^{-1} \sum_{i=1}^n \delta_i$ and diffusion coefficient $1/n$ and is independent of the vector of spacings $(X_{(i)}(t) - X_{(i+1)}(t), 1 \leq i \leq n-1, t \geq 0)$.
- (ii) Let β be an independent one-dimensional Brownian motion ($\beta_0 = 1$) with a negative drift -1 which is reflected at the origin. Then the process defined by

$$(43) \quad Y_i(t) = X_{(i)}(t) - \bar{X}(t) + \frac{\beta(t)}{\sqrt{n}}, \quad i = 1, 2, \dots, n, \quad t \geq 0,$$

is an n -dimensional Brownian motion with constant drift and identity covariance matrix which is normally reflected in the wedge $G = \cap_{i=1}^n G_i$, where

$$(44) \quad G_i = \{x_i - x_{i+1} \geq 0\}, \quad 1 \leq i \leq n-1, \quad \text{and} \quad G_n = \left\{ \sum_i x_i \geq 0 \right\}.$$

Proof. The proof of the first assertions can be found in Lemma 7 in [37]. The argument for the second assertion uses Lemma 16. Please see [37, p. 2188]. \square

Notice that the spacing vector between the Y_i 's and that between the X_i 's are the same, i.e.,

$$Y_i - Y_{i+1} \equiv X_{(i)} - X_{(i+1)}, \quad \text{for all } i.$$

Thus, the local time at zero for every $Y_i - Y_{i+1}$ by $\sqrt{2}$ is exactly $L_{i,i+1}/2$, a fact that we use below. The advantage of considering Y_i 's is that now we can use Theorem 14.

Theorem 18. *For any choice of parameters $(\delta_1, \dots, \delta_n)$, the vector of increasing random processes*

$$L_{j,j+1}(t), \quad j = 1, 2, \dots, n-1, \quad 0 \leq t \leq T,$$

satisfies the QTCI with respect to the metric

$$\hat{d}(\omega, \omega') = \sup_{0 \leq t \leq T} \sqrt{\frac{1}{n} \sum_{i=1}^n (\omega_i(t) - \omega'_i(t))^2}.$$

with a constant Cn^6T where C is a universal constant.

Remark. Let us point out to the reader that a better bound of n^5T has been obtained in the follow-up article by Pal and Shkolnikov [38]. We strongly believe that this is optimal although we cannot prove it.

Proof. Consider the Skorokhod map, described in Definition 1, on the polyhedral domain G given in (44). As in Theorem 14, we can take the vectors

$$\eta(i) = \frac{e(i) - e(i+1)}{\sqrt{2}}, \quad \text{and} \quad d(i) = \eta(i), \quad \text{for each } i = 1, 2, \dots, n-1.$$

Here the $e(i)$'s represent the standard basis vector in \mathbb{R}^n . For $i = n$, we need to take $\eta(n) = n^{-1/2}\mathbf{1}$, and $d(n) = \eta(n)$, where $\mathbf{1}$ is the vector of all ones.

Then the $d(i)$'s are linearly independent and satisfies $\langle d(i), \eta(i) \rangle = 1$. Define the $n \times n$ matrix D by

$$D = [d(1) \mid d(2) \mid \dots \mid d(n)].$$

Then D has full rank. The matrix Q in (37) is then given

$$(45) \quad Q = [q_{ij}] = I - D^*D = \begin{cases} \frac{1}{2}1\{|i-j|=1\}, & \text{if } 1 \leq i, j \leq n-1, \\ 0, & \text{if } i=j, i=n, \text{ or } j=n. \end{cases}$$

Here D^* represents the matrix transpose of D .

It is easy to see that Q is a submatrix of a stochastic transition matrix. That is, consider P to be the transition probability matrix of a simple symmetric random walk on the integers $\{0, 1, 2, \dots, n\}$, with absorbing boundary conditions. We then *identify* states 0 and n by changing the P matrix to have $P(n-1, 0) = 1/2 = 1 - P(n-1, n)$ and $P(n, 0) = 1 = 1 - P(n, n)$. Then Q is the submatrix of transition probabilities corresponding to states $\{1, 2, \dots, n\}$. Since these states are transient, it follows that $\sigma(Q) < 1$. Thus, by Theorem 14, the Skorokhod map on this domain is Lipschitz. We need to estimate the Lipschitz coefficient.

To do this, we need to find the set B in Assumption 2.1 (Set B) in [12, p. 160]. We follow the notation used in [12, 13]. Let $\text{SP}(d(i), \eta(i), 0)$ denote the Skorokhod problem in our domain G . Consider another Skorokhod problem $\text{SP}(e(i), D^*\eta(i), 0)$, where D^* refers to the transpose of D . As noted in [13, p. 203] (and easily verifiable), the matrix Q corresponding to this problem is the same as the Q in (45). Thus the Skorokhod map is again Lipschitz.

Let \hat{B} be the set satisfying (Assumption 2.1 [12, p. 160]) for the $\text{SP}(e(i), D^*\eta(i), 0)$ for some $\delta > 0$. Then, as mentioned in [13, p. 202], $B = D\hat{B}$ satisfies Assumption 2.1 for the original problem $\text{SP}(d(i), \eta(i), 0)$.

We claim the following description of the set \hat{B} . Consider a vector u such that

$$(46) \quad Qu < u, \quad u_i > 0, \quad i = 1, 2, \dots, n.$$

Then, we claim that

$$(47) \quad \hat{B} = \{x \in \mathbb{R}^n : |x_i| \leq u_i\}, \quad \delta = \min_i (u_i - (Qu)_i).$$

Let us verify the conditions of Assumption 2.1 in [12]. Consider a boundary point $z \in \partial\hat{B}$. Then, without loss of generality (and considering only points in the relative interior), we can assume that $z_i = u_i$, for some i , and $|z_j| \leq u_j$ for all $j \neq i$. In that case the inward normal to z is $-e(i)$.

For our choice of δ , assume that, for some j , one has

$$|\langle z, D^*\eta(j) \rangle| = |\langle z, (I - Q)e(j) \rangle| < \delta, \quad \text{since } \eta(j) = d(j) = De(j).$$

Now, if $j = i$, then

$$|\langle z, (I - Q)e(j) \rangle| = \left| u_i - \frac{z_{i+1} + z_{i-1}}{2} \right| \geq u_i - \frac{u_{i+1} + u_{i-1}}{2} = u_i - (Qu)_i \geq \delta,$$

since every u_j is positive, $|z_j| \leq u_j$, and $Qu < u$. Thus, $|\langle z, D^*\eta(j) \rangle| < \delta$ implies $j \neq i$, and hence $\langle \nu, e(j) \rangle = -\langle e(i), e(j) \rangle = 0$. This verifies [12, eqn. (2.2)] and establishes our claim.

It is straightforward to see that the same δ works for the set $B = D\hat{B}$ as well. To wit, take any point $y \in \partial B$. Then, $y = Dz$, for some $z \in \hat{B}$. Thus $\langle y, \eta(j) \rangle = \langle z, D^*\eta(j) \rangle < \delta$ implies $-\langle \nu, e(j) \rangle = 0$, where ν , as before, is the normal vector at z . But the normal vector at y , ν_y , is obviously $(D^{-1})^*\nu$. Thus $\langle \nu_y, d(j) \rangle = \langle \nu, D^{-1}d(j) \rangle = \langle \nu, e(j) \rangle = 0$. This satisfies Assumption 2.1 in [12].

Thus to find the Lipschitz coefficient we need to find a vector u satisfying (46). Let $v(x)$ be any nonnegative strictly concave function on $[0, 1]$ such that $v(0) = 0$.

Define $u(x) = v(x/n)$ for $0 \leq x \leq n$, and let $u_k = v(k/n)$, for $k = 1, 2, \dots, n$. For example, we choose $v(x) = x(1-x)$. Then

$$v''(x) = -2, \quad 0 < x < 1.$$

Then, by the strict concavity of v it follows that

$$u_k > \frac{1}{2}(u_{k-1} + u_{k+1}), \quad k = 1, 2, \dots, n-1.$$

Hence, it follows that $(Qu)_i < u_i$ for every $i = 1, 2, \dots, n-1$. For $i = n$ the inequality is not strict. However, this is easy to correct since the row Q_{n*} is the zero row and we can choose u_n to be any small enough positive number.

Now, for our choice of v , it follows that

$$\delta = \min_i (u_i - (Qu)_i) \geq \frac{1}{2} \inf_{0 \leq x \leq n} -u''(x) = \frac{1}{2n^2} \inf_{0 \leq x \leq 1} -v''(x) = \frac{1}{n^2}.$$

In fact, to simplify, we define $\delta = n^{-2}$ for the rest of the analysis.

Now to find what the Lipschitz constant is for the Skorokhod problem on the wedge G we use the Remark made in page 161 in [12]. If the set B constructed above satisfies Assumption 2.1 in [12] for some $\delta > 0$, then one plus the diameter of $\delta^{-1}B$ serves as the Lipschitz constant for the Skorokhod map. We already know what δ is. Let us now estimate the diameter of the set B in (47). Since extreme points are preserved under linear transforms, we get

$$\text{diam}(B) \leq 2 \max_{\sigma} \left\| \sum_{i=1}^n \sigma_i u_i d(i) \right\|,$$

where the maximum is running over all choices of coefficients $\sigma_i = \pm 1$.

We claim that the above diameter is of the order \sqrt{n} . This is easy to see since each $d(i)$ has norm one and their inner products are given by the entries of the matrix D^*D . Thus by expanding the vector $\sum_{i=1}^n \sigma_i u_i d(i)$ for any choice of σ we get

$$\begin{aligned} \left\| \sum_{i=1}^n \sigma_i u_i d(i) \right\|^2 &= \sum_{i=1}^n u_i^2 - \sum_{i=1}^{n-1} \sigma_i \sigma_{i-1} u_i u_{i-1} \\ &\leq \sum_{i=1}^n u_i^2 + \frac{1}{2} \sum_{i=1}^{n-1} (u_{i+1}^2 + u_i^2) \leq 3 \sum_{i=1}^n u_i^2. \end{aligned}$$

We now try to bound. By our choice of the functions v and u we get

$$\sum_{i=1}^n u_i^2 = \sum_{i=1}^n v^2\left(\frac{i}{n}\right) \sim n \int_0^1 x^2(1-x)^2 dx = \frac{n}{30}.$$

In other words,

$$\text{diam}(B) \leq 2 \max_{\sigma} \left\| \sum_{i=1}^n \sigma_i u_i d(i) \right\| \leq 4\sqrt{n}.$$

Combining with our estimate of δ we get that the diameter of $\delta^{-1}B$ is at most $4n^{5/2}$. This, as argued, previously, serves as our Lipschitz constant for the Skorokhod map on G as defined in (44).

Now consider the RBM $Y(t)$ described in Lemma 17. The i th coordinate process $Y_i(t)$ has a semimartingale decomposition

$$Y_i(t) = M_i(t) + \xi_i(t) + \tilde{\eta}_i(t),$$

where M_i is a martingale and $\xi_i(t)$, $\tilde{\eta}_i$ are of finite variation, the former being the absolutely continuous part (the drift) and the latter being the component that is mutually singular with respect to the Lebesgue measure (the local time). Comparing expressions (40) with (43) we get

$$\tilde{\eta}_i(t) = \frac{1}{\sqrt{2}} (L_{i,i+1}(t) - L_{i-1,i}(t)) + \frac{1}{\sqrt{n}} L_0(t), \quad L_{0,1} \equiv L_{n,n+1} \equiv 0.$$

Here $L_0(t)$ is the local time at zero for the Brownian motion $\beta(t)$. The drift is a constant $\delta_i - 1/\sqrt{n}$, and the martingale M_i is a Brownian motion, independent of all the other M_j 's.

Thus in Definition 1 one can take the driving noise, ψ , to be a typical path of the n -dimensional Brownian motion with constant drifts $(M_i + \xi_i, i = 1, 2, \dots, n)$. Then, according to Theorem 6, this multidimensional noise, during the time interval $0 \leq t \leq T$, satisfies QTCI with a constant $4n^{-1}T$.

We will now use Lemma 4. We take the metric space (E, d_E) to be $C^n[0, T]$ with the \bar{d} metric, and take μ to be the multidimensional Wiener measure. We also take (F, d_F) to be $C^n[0, T]$ with the metric

$$\hat{d}(\omega, \omega') = \sup_{0 \leq t \leq T} \sqrt{\frac{1}{n} \sum_{i=1}^n (\omega_i(t) - \omega'_i(t))^2}.$$

By (36), the map that takes ψ to $\tilde{\eta}$ is Lipschitz. From what we have done so far, it is now clear that the map that takes $M_i + \xi_i$ to $\tilde{\eta}$ induces a Lipschitz map from (E, d_E) to (F, d_F) that is Lipschitz with a Lipschitz constant $O(n^{5/2})$. Since multidimensional Brownian motions satisfy QTCI with a constant $4n^{-1}T$, this shows that the process $\tilde{\eta}$ satisfies QTCI with a constant $O(n^4)$.

Now consider the map that takes $\tilde{\eta}$ to the vector L . We claim that it is Lipschitz with a constant which is of order \sqrt{n} . To see this, note that the map that takes L to $\tilde{\eta}$ is linear:

$$\tilde{\eta} - \frac{L_0}{\sqrt{n}} \mathbf{1} = \frac{1}{\sqrt{2}} [e(1) - e(2) \mid e(2) - e(3) \mid \dots \mid e(n-1) - e(n)] L$$

We claim that the smallest eigenvalue of the $(n-1) \times n$ dimensional matrix

$$S = \frac{1}{\sqrt{2}} [e(1) - e(2) \mid e(2) - e(3) \mid \dots \mid e(n-1) - e(n)].$$

is of the order $1/\sqrt{n}$.

But it is easy compute inner products between columns of S . Thus, for any vector $v \in \mathbb{R}^{n-1}$, we get

$$\|Sv\|^2 = \sum_{i=1}^{n-1} v_i^2 - \sum_{i=2}^{n-1} v_i v_{i-1} \geq \sum_{i=1}^{n-1} v_i^2 - \frac{1}{2} \sum_{i=2}^{n-1} (v_i^2 + v_{i-1}^2) = \frac{v_1^2 + v_{n-1}^2}{2}.$$

The equality above is achieved when $v_1 = v_2 = \dots = v_{n-1}$. Thus, under the constraint $\sum_i v_i^2 = 1$, the infimum of the above norm square is $1/n$. This shows that the map that takes $\tilde{\eta}$ to L is Lipschitz with a constant $O(1/\sqrt{n})$.

Thus the map that takes $M + \xi$ to the vector of local times L induces a Lipschitz map with a Lipschitz constant $O(n^3)$. By Lemma 4 it follows then that the law of the process $(\eta_1, \dots, \eta_{n-1})$ satisfies QTCI with a constant which is of the order of $n^5 T$. This completes the proof of the Proposition. \square

Finally we prove Theorem 1.

Proof of Theorem 1. This proof is immediate once we note that the maximum function is Lipschitz with respect to the \hat{d} norm required in Theorem 18. \square

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